

AN EXTENSION OF THE BAIRE PROPERTY

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ABSTRACT. Here we define the notion of residually null sets, a σ -ideal of subsets of a Polish space which contains the meager sets and, in some contexts, generalizes the notion of universally null. From this σ -ideal we realize a σ -algebra of sets which is consistently a strict extension of the Baire property algebra. We then uncover a generalization to Pettis' Theorem which furnishes a new automatic continuity result.

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1. INTRODUCTION

The notion of a subset of X having probability zero with respect to a fixed collection of probability measures is, in general, a rather useless one in cases in which X is large (infinite-dimensional or not locally compact). For example, when X is a Polish space, every Borel probability measure is supported on a σ -compact set. Possible exceptions of interest are measures motivated by physical considerations (e.g., Brownian motion) or invariance properties (Haar measure and related measures on locally compact groups or quotient spaces). Heuristically, we will want a subset of X to be “small” provided that it is null with respect to “most” probability measures on X . How can we make this notion of “most” probability measures mathematically precise? We can topologize the space of probability measures in a Polish way, provided X is Polish. We can then choose a probability measure on the space of measures on X . The objection to this is that there is no natural choice for such a probability measure and, if X is large and therefore the space of measures is large, such a probability measure is supported on

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a very thin subset of the space of probability measures on X . We therefore turn to notions of category, a more natural and topologically invariant concept.

Let X be a Polish space and $\mathcal{M}(X)$ be the collection of all Borel probability measures on X . Let $C_b(X)$ be the collection of all functions $f : X \rightarrow \mathbb{R}$ which are continuous and bounded. Endow $\mathcal{M}(X)$ with the coarsest topology for which each map

$$\mu \mapsto \int f d\mu, \mathcal{M}(X) \rightarrow \mathbb{R},$$

where $f \in C_b(X)$, is continuous. With this topology, $\mathcal{M}(X)$ is Polish by [12, Theorem 6.5] and the set of continuous Borel probability measures $\mathcal{M}_c(X)$ is a G_δ subset of $\mathcal{M}(X)$ by [12, Theorem 8.1]. Moreover, with [12, Corollary 8.1], $\mathcal{M}_c(X)$ is dense if and only if X has no isolated points. From this, we see that we can apply results from the theory of Polish spaces to $\mathcal{M}(X)$ or $\mathcal{M}_c(X)$.

2. RESIDUALLY NULL SETS

Definition 1. Let X be Polish. Define $\mathcal{N} : \wp(X) \rightarrow \wp(\mathcal{M}(X))$ by the rule

$$\mathcal{N}(A) = \{\mu \in \mathcal{M}(X) : \mu^*(A) = 0\}$$

where μ^* is the outer measure induced by μ . We will call $\mathcal{N}(A)$ the *annihilator* of A .

Definition 2. Let X be a Polish space and let $A \subseteq X$. We say that A is *residually measurable* if A is μ -measurable for a co-meager set of $\mu \in \mathcal{M}(X)$, A is *residually null* if $\mathcal{N}(A)$ is co-meager in $\mathcal{M}(X)$, or A is *co-residually null* if $X \setminus A$ is residually null.

To witness the fact that the collection of residually null subsets of a Polish space form a σ -ideal, consider the following immediate facts about the annihilator.

Lemma 3. Let X be Polish. If $A \subseteq B \subseteq X$, then $\mathcal{N}(B) \subseteq \mathcal{N}(A)$. Also, for a countable family \mathcal{E} of subsets of X , $\mathcal{N}(\bigcup \mathcal{E}) = \bigcap \{\mathcal{N}(E) : E \in \mathcal{E}\}$.

Recall the following notions.

Definition 4. Let X be a Polish space and $A \subseteq X$. We say that A is *universally measurable* if A is μ -measurable (in the sense of Carathéodory) for every $\mu \in \mathcal{M}(X)$. Similarly, we say that A is *universally null* if $\mu^*(A) = 0$ for every $\mu \in \mathcal{M}_c(X)$.

Immediately, all universally measurable sets are residually measurable. Moreover, if X is a Polish space with no isolated points and A is universally

null, then A is residually null. So, in spaces of usual interest, like \mathbb{R} , we see that the σ -ideal of residually null sets is finer than the σ -ideal of universally null sets. In fact, it is strictly finer but we will need Theorem 9 to witness this fact.

Proposition 5. The notion of being residually null is a topological invariant.

3. COMPLEXITY CONSIDERATIONS

Here, we turn our attention to the descriptive relationships between sets and their annihilators. To begin, recall

Theorem 6 (Portmanteau). Let X be a Polish space, $\{\mu_n : n \in \omega\} \subseteq \mathcal{M}(X)$, and $\mu \in \mathcal{M}(X)$. Then the following are equivalent:

- $\mu_n \rightarrow \mu$;
- for every closed set $F \subseteq X$, $\limsup\{\mu_n(F) : n \in \omega\} \leq \mu(F)$;
- for every open set $U \subseteq X$, $\mu(U) \leq \liminf\{\mu_n(U) : n \in \omega\}$.

The Portmanteau Theorem actually has some deeper consequences for the Borel structure of certain sets of measures.

Proposition 7. Let X be Polish, $B \subseteq X$ be Borel, and $\varepsilon \in [0, 1]$. Then

$$M(B, \varepsilon) := \{\mu \in \mathcal{M}(X) : \mu(B) \leq \varepsilon\}$$

is Borel. In particular, if $B \in \Sigma_\alpha^0(X)$, then $M(B, \varepsilon) \in \Pi_\alpha^0(\mathcal{M}(X))$ and, if $B \in \Pi_\alpha^0(X)$, then $M(B, \varepsilon) \in \Pi_{\alpha+1}^0(\mathcal{M}(X))$.

Proof. Let $M^*(B, \varepsilon) = \{\mu \in \mathcal{M}(X) : \mu(B) < \varepsilon\}$. Observe that

$$(1) \quad M^*(B, \varepsilon) = \bigcup \{M(B, \varepsilon - 2^{-k}) : k \in \omega\}.$$

Suppose U is open. Then, by the Portmanteau Theorem, $M(U, \varepsilon)$ is closed. Hence, by (1), $M^*(U, \varepsilon)$ is an F_σ .

Now, suppose F is closed. Notice that

$$\mu(F) \leq \varepsilon \iff 1 - \varepsilon \leq \mu(X \setminus F).$$

Hence, we see that

$$M(F, \varepsilon) = \mathcal{M}(X) \setminus M^*(X \setminus F, 1 - \varepsilon),$$

which provides us with the fact that $M(F, \varepsilon)$ is a G_δ .

Now, let α be any countable ordinal bigger than 1 and suppose, inductively, that for each $\beta < \alpha$, the following hold:

- $(\forall B \in \Sigma_\beta^0(X))(M(B, \varepsilon) \in \Pi_\beta^0(\mathcal{M}(X)) \text{ and } M^*(B, \varepsilon) \in \Sigma_{\beta+1}^0(\mathcal{M}(X)))$;
- $(\forall B \in \Pi_\beta^0(X))(M(B, \varepsilon) \in \Pi_{\beta+1}^0(\mathcal{M}(X)))$.

Let $B \in \Sigma_\alpha^0(X)$ and pick $\{B_k : k \in \omega\} \subseteq \bigcup\{\Pi_\beta^0(X) : \beta < \alpha\}$ so that $B = \bigcup\{B_k : k \in \omega\}$. For each $k \in \omega$, let $\beta_k < \alpha$ be so that $B_k \in \Pi_{\beta_k}^0(X)$ and then define $\gamma_k = \max\{\beta_i : i \leq k\} < \alpha$. It follows that $B_0 \cup B_1 \cup \dots \cup B_k \in \Pi_{\gamma_k}^0(X)$ since $\Pi_{\gamma_k}^0(X)$ is closed under finite unions. So

$$M(B_0 \cup B_1 \cup \dots \cup B_k, \varepsilon) \in \Pi_{\gamma_k+1}^0(\mathcal{M}(X)) \subseteq \Pi_\alpha^0(\mathcal{M}(X))$$

by our inductive hypothesis.

As $\Pi_\alpha^0(\mathcal{M}(X))$ is closed under countable intersections,

$$M(B, \varepsilon) = \bigcap \{M(B_0 \cup B_1 \cup \dots \cup B_k, \varepsilon) : k \in \omega\} \in \Pi_\alpha^0(\mathcal{M}(X)).$$

From (1), we see that $M^*(B, \varepsilon) \in \Sigma_{\alpha+1}^0(\mathcal{M}(X))$.

Now, let $B \in \Pi_\alpha^0(X)$ and notice that

$$M(B, \varepsilon) = \mathcal{M}(X) \setminus M^*(X \setminus B, 1 - \varepsilon),$$

Since $X \setminus B \in \Sigma_\alpha^0(X)$ and $M^*(X \setminus B, 1 - \varepsilon) \in \Sigma_{\alpha+1}^0(\mathcal{M}(X))$ by the above calculation, we see that $M(B, \varepsilon) \in \Pi_{\alpha+1}^0(\mathcal{M}(X))$. \square

Corollary 8. Let X be Polish and $B \subseteq X$ be Borel. Then $\mathcal{N}(B)$ is Borel. In particular, if $B \in \Sigma_\alpha^0(X)$, $\mathcal{N}(B) \in \Pi_\alpha^0(\mathcal{M}(X))$ and, if $B \in \Pi_\alpha^0(X)$, then $\mathcal{N}(B) \in \Pi_{\alpha+1}^0(\mathcal{M}(X))$.

The following appears as [3, 3.16] in the context of compact metrizable spaces.

Theorem 9. Let X be a Polish space. If a set $A \subseteq X$ is meager, then A is residually null.

Proof. By Lemma 3, it suffices to consider the case when A is closed and nowhere dense. Let $U = X \setminus A$ and notice that U is open and dense in X . By [12, Lemma 6.1], we see that $\mathcal{N}(A)$ is dense and, by Corollary 8, we have that $\mathcal{N}(A)$ is a G_δ , finishing the proof. \square

Now we are sufficiently equipped to see that the class of residually null sets is strictly finer than the class of universally null sets. Notice that the middle-thirds Cantor set is closed and nowhere dense but not universally null as any continuous non-zero Borel probability on 2^ω can be extended to a continuous Borel probability on \mathbb{R} .

It also turns out that the class of residually measurable sets is strictly finer than the class of universally measurable sets. Let $A \subseteq [0, 1]$ be non-measurable with respect to the Lebesgue measure. Then define $E = A \times \{0\} \subseteq [0, 1]^2$. It follows that E is a residually measurable subset of $[0, 1]^2$ since it's meager but not universally null since we can extend the Lebesgue measure on $[0, 1]$ to a continuous Borel probability on $[0, 1]^2$ in the obvious way.

Lemma 10. Let X be a Polish space, $\mu \in \mathcal{M}(X)$, and $A, B \subseteq X$ so that $\mu^*(A \Delta B) = 0$. If A is μ -measurable, B is μ -measurable.

Corollary 11. Let X be a Polish space. If $A \subseteq X$ has the Baire property, then A is residually measurable.

Proof. Let $G \subseteq X$ be a G_δ and $M \subseteq X$ be meager so that $A = G \cup M$. Theorem 9 guarantees that $\mathcal{N}(M)$ is co-meager in $\mathcal{M}(X)$. For any $\mu \in \mathcal{N}(M)$, we have that $\mu^*(A \Delta G) \leq \mu^*(M) = 0$. So, by Lemma 10, we see that A is μ -measurable for each $\mu \in \mathcal{N}(M)$ which establishes that A is residually measurable. \square

Corollary 12. Let X be a Polish space. If $A \subseteq X$ has the Baire property, then $\mathcal{N}(A)$ has the Baire property.

Proof. Let $G \subseteq X$ be a G_δ and $M \subseteq X$ be meager so that $A = G \cup M$. Then, notice that $\mathcal{N}(A) = \mathcal{N}(G) \cap \mathcal{N}(M)$. Corollary 8 informs us that $\mathcal{N}(G)$ is Borel and Theorem 9 tells us that $\mathcal{N}(M)$ is co-meager so we conclude that $\mathcal{N}(A)$ is a set with the Baire property. \square

Lemma 13. Let X be Polish and $U \subseteq X$ be open and non-empty. Then $\mathcal{N}(U)$ is closed and nowhere dense in $\mathcal{M}(X)$. That is, no non-empty open set is residually null.

Proof. By the Portmanteau Theorem, we see that $\{\mu \in \mathcal{M}(X) : \mu(U) > 0\}$ is open. To see that it is dense, let $\mu \in \mathcal{M}(X)$ be arbitrary and notice that, for $x \in U$ and $\lambda \in [0, 1]$, the measure $\nu_\lambda := (1 - \lambda)\delta_x + \lambda\mu$ gives U positive measure. Finally, notice that $\nu_\lambda \rightarrow \mu$ as $\lambda \rightarrow 1$. This completes the proof. \square

The following partial converse to Theorem 9 appears as [3, Theorem 3.17], again, in the context of compact metric spaces.

Theorem 14. Let X be Polish and $A \subseteq X$ be a residually null set with the Baire property. Then A is meager.

Proof. As A has the Baire property, let U be open so that $U \Delta A$ is meager, and suppose, by way of contradiction, that $U \neq \emptyset$. Since $\mathcal{N}(A \Delta U)$ is co-meager by Theorem 9, Lemma 13 provides us with the fact that

$$G := \mathcal{N}(A \Delta U) \cap \mathcal{N}(A) \cap \{\mu \in \mathcal{M}(X) : \mu(U) > 0\}$$

is co-meager in $\mathcal{M}(X)$. Let $\mu \in G$ and notice that

$$0 < \mu(U) \leq \mu((U \setminus A) \cup A) \leq \mu(U \setminus A) + \mu(A) \leq \mu(U \Delta A) = 0,$$

a contradiction. Hence, $U = \emptyset$ and we see that A is meager. \square

R. Solovay demonstrated, in [15], that it is consistent with ZF+DC (ZF and Dependent Choice) that all sets of reals have the Baire property assuming the existence of an inaccessible cardinal. Later, S. Shelah, in [14], showed that the inaccessible is unnecessary to obtain a model of ZF+DC where every set of reals has the Baire property. Hence, it is consistent with ZF+DC that all residually null sets are meager by Theorem 14. Since DC is strictly weaker than the full Axiom of Choice, we are led to the natural question:

Question 1. Is it consistent with ZFC that every residually null set is meager?

An answer to this would answer a question of [9], revisited as unresolved in [8], which asks if it is consistent with ZFC that all universally measurable sets have the Baire property. We will shortly elaborate more on this.

Hitherto, we've concluded complexity limitations on the annihilator of a set A assuming some complexity limitations on A . Now, we'll show that complexity bounds on the annihilator of A can impose other complexity bounds upon A itself.

Toward such an end, combining [12, Lemma 6.1] and [12, Lemma 6.2], we obtain that

Lemma 15. For any Polish space X , the map $x \mapsto \delta_x$, $X \rightarrow \mathcal{M}(X)$, is a homeomorphism onto its range. Moreover, $\{\delta_x : x \in X\}$ is a closed subspace of $\mathcal{M}(X)$.

Proposition 16. Let X be a Polish space and suppose Γ is one of the classes Σ_α^i or Π_α^i where $i = 0, 1$ and $\alpha \geq 1$ is a countable ordinal. Then, if $\mathcal{N}(A) \in \Gamma(\mathcal{M}(X))$, $A \in \neg\Gamma(X)$.

Proof. Suppose that $\mathcal{N}(A) \in \Gamma(\mathcal{M}(X))$. Notice that

$$\mathcal{N}(A) \cap \{\delta_x : x \in X\} = \{\delta_x : x \in X \setminus A\}.$$

By Lemma 15, $\{\delta_x : x \in X \setminus A\} \in \Gamma(X)$. It follows that $A \in \neg\Gamma(X)$. \square

Proposition 17. Suppose A is residually null and that $\mathcal{N}(A)$ is either analytic or co-analytic. Then A is meager.

Proof. Recall that analytic sets and co-analytic sets have the Baire property. Then, observe that, by Proposition 16, A is analytic or co-analytic. Therefore, Theorem 14 guarantees that A is meager. \square

Corollary 18. For any Polish space X , a set $A \subseteq X$ is Borel if and only if $\mathcal{N}(A)$ is Borel.

Proof. Apply Corollary 8 and Proposition 16. \square

In sum, it seems that the descriptive complexity of A is intimately related to the descriptive complexity of its annihilator. Inspired by Corollaries 12 and 18, one could ask if it is consistent with ZFC that A have the Baire property whenever its annihilator has the Baire property. Alas, this is a more general version of Question 1 and we will now see that a negative answer is consistent.

Sets which are universally null but lack the Baire property began to bubble up into the mathematical consciousness more than a century ago. Assuming the Continuum Hypothesis, both N. Luzin and P. Mahlo independently constructed a set, now called a Luzin set, of power continuum which has countable intersection with each meager set. About two decades later, combined works of W. Sierpiński and E. Szpilrajn demonstrated that a Luzin set is actually universally null. Since a Luzin set is non-meager and universally null, Theorem 14 informs us that a Luzin set cannot have the Baire property. For a more detailed discussion of this along with the relevant references, see [10].

As it turns out, the assumption of CH to build such sets is not absolutely necessary. In fact, [8] provides a non-meager set which is universally null under more general conditions. Let \mathcal{M} be the class of all meager subsets of \mathbb{R} . Recall that $\text{cov}(\mathcal{M})$ is the least cardinal κ so that \mathbb{R} is covered by a union of κ -many meager sets and that $\text{cof}(\mathcal{M})$ is the least cardinal κ so that there exists a family \mathcal{F} of cardinality κ so that every meager set is contained in a member of \mathcal{F} . Then, [8, Theorem 4.3] informs us that, under the assumption $\text{cov}(\mathcal{M}) = \text{cof}(\mathcal{M})$, there exists a universally null set which does not have the Baire property. The condition $\text{cov}(\mathcal{M}) = \text{cof}(\mathcal{M})$ is implied by both Martin's Axiom and, less generally, the Continuum Hypothesis.

4. EXTENDING THE BAIRE PROPERTY

Definition 19. Let X be a Polish space. We say that $A \subseteq X$ has the *Residually Null Baire Property* or that A is an \mathcal{R} -set if there exists an open set $U \subseteq X$ so that $U \triangle A$ is residually null. We will let $\mathcal{R}(X)$ be the collection of all subset of X which are \mathcal{R} -sets.

Proposition 20. The property of being an \mathcal{R} -set is a topological invariant.

Proof. Let $h : X \rightarrow Y$ be a homeomorphism where X and Y are Polish spaces. Suppose $A \in \mathcal{R}(X)$ and pick U open so that $A \triangle U$ is residually null. Notice that $h[U]$ is open and that

$$h[A] \triangle h[U] = h[A \triangle U].$$

Since the class of residually null sets is invariant under homeomorphisms by Proposition 5, we see that $h[A] \in \mathcal{R}(Y)$. \square

Lemma 21. Let $A, B, C, A_\xi, B_\xi \subseteq X$ for $\xi < \kappa$ where κ is any cardinal. Then

- $A^c \triangle B^c = A \triangle B$;
- $A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$;
- $(A \cap B) \triangle (C \cap D) \subseteq (A \triangle C) \cup (B \triangle D)$;
- $[\bigcup \{A_\xi : \xi < \kappa\}] \triangle [\bigcup \{B_\xi : \xi < \kappa\}] \subseteq \bigcup \{A_\xi \triangle B_\xi : \xi < \kappa\}$.

Theorem 22. For a Polish space X , the collection $\mathcal{R}(X)$ is a σ -algebra and is the smallest σ -algebra containing the open sets and the residually null sets.

Proof. First, we show that the open sets and the residually null sets are in $\mathcal{R}(X)$. If U is open, then $U \triangle U = \emptyset$ so $U \in \mathcal{R}(X)$. If A is residually null, then $A \triangle \emptyset = A$ so we see that $A \in \mathcal{R}(X)$.

Now we will see that $\mathcal{R}(X)$ is a σ -algebra. Suppose $A \in \mathcal{R}(X)$ and let U be open so that $A \triangle U$ is residually null. Then

$$A^c \triangle \text{int}_X(U^c) \subseteq (A^c \triangle U^c) \cup (U^c \triangle \text{int}_X(U^c)) = (A \triangle U) \cup (U^c \setminus \text{int}_X(U^c)).$$

Since U is open, $U^c \setminus \text{int}_X(U^c)$ is closed and nowhere dense. Hence, $A^c \in \mathcal{R}(X)$. Now, suppose $\{A_n : n \in \omega\} \subseteq \mathcal{R}(X)$ and let U_n be open so that $A_n \triangle U_n$ is residually null for each $n \in \omega$. Then, observe that

$$\left[\bigcup \{A_n : n \in \omega\} \right] \triangle \left[\bigcup \{U_n : n \in \omega\} \right] \subseteq \bigcup \{A_n \triangle U_n : n \in \omega\},$$

providing us with the fact that $\bigcup \{A_n : n \in \omega\} \in \mathcal{R}(X)$.

Now suppose \mathcal{S} is any σ -algebra of sets of X containing both the open sets and the residually null sets. Let $A \in \mathcal{R}(X)$ and let U be open and R be residually null so that $A \triangle U = R$. Since $U, R \in \mathcal{S}$, we see that $A = U \triangle R \in \mathcal{S}$. \square

An immediate observation from Theorems 9 and 22 is that $\mathcal{BP}(X) \subseteq \mathcal{R}(X)$. Also, all residually null sets are \mathcal{R} -sets which gives us that, by discussion above, the containment $\mathcal{BP}(X) \subseteq \mathcal{R}(X)$ is proper in all models of ZFC+MA or, less generally, ZFC+CH.

Lemma 23. Let X be a Polish space and $A \subseteq X$. Then the following are equivalent:

- (i) A is an \mathcal{R} -set;
- (ii) There exists an \mathcal{R} -set B so that $A \triangle B$ is residually null;
- (iii) $A = (U \setminus N) \cup M$ where U is open and N and M are both residually null.

Proof. ((i) \Rightarrow (ii)) By definition, if A is an \mathcal{R} -set, there exists an open set U so that $A \triangle U$ is residually null and U is an \mathcal{R} -set.

((ii) \Rightarrow (i)) Let B be an \mathcal{R} -set so that $A \triangle B =: N$ is residually null. Then, since $A = B \triangle N$ and the family of \mathcal{R} -sets is a σ -algebra by Theorem 22, we see that A is an \mathcal{R} -set.

((i) \Rightarrow (iii)) Let U be open so that $A \triangle U =: N$ is residually null. Then notice that, letting $M = N \setminus U$,

$$A = U \triangle N = (U \setminus N) \cup (N \setminus U) = (U \setminus N) \cup M,$$

the desired form.

((iii) \Rightarrow (i)) This follows immediately from Theorem 22. \square

Theorem 24. All \mathcal{R} -sets are residually measurable.

Proof. Suppose $A \in \mathcal{R}(X)$ and let U be open so that $U \triangle A$ is residually null. For any $\mu \in \mathcal{N}(U \triangle A)$, we see that A is μ -measurable by Lemma 10. Therefore, A is residually measurable. \square

Definition 25. For a Polish space X , define $R : \wp(X) \rightarrow \wp(X)$ by

$$R(A) = \bigcup \{U \text{ open} : U \cap A \text{ is residually null}\}.$$

Notice that $R(A)$ is open. Define $CR(A) = X \setminus R(A)$ and $ICR(A) = \text{int}_X(CR(A))$. Recall that, inspired by the set function D appearing in [7, p. 83],

$$M(A) := \bigcup \{U \text{ open} : U \cap A \text{ is meager}\}$$

satisfies $M(A) \subseteq R(A)$ as all meager sets are residually null. From $D(A) := X \setminus M(A)$, we see immediately that $CR(A) \subseteq D(A)$.

Theorem 26. Let X be a Polish space. For any set $A \subseteq X$, both $A \cap R(A)$ and $A \cap \text{cl}_X(R(A))$ are residually null.

Proof. Let \mathcal{B} be a countable base for the topology on X and let

$$\mathcal{U} = \{B \in \mathcal{B} : B \cap A \text{ is residually null}\}.$$

Enumerate $\mathcal{U} = \{U_n : n \in \omega\}$ and define $U = \bigcup \mathcal{U}$. Surely, $U \subseteq R(A)$.

Suppose V is open so that $V \cap A$ is residually null. As

$$V = \bigcup \{B \in \mathcal{B} : B \subseteq V\}$$

and every $B \in \mathcal{B}$ so that $B \subseteq V$ satisfies the property that $A \cap B \subseteq A \cap V$ is residually null, we see that $V \subseteq U$. Hence, $R(A) \subseteq U$ which establishes that $U = R(A)$.

Now, notice that

$$A \cap R(A) = A \cap \bigcup \{U_n : n \in \omega\} = \bigcup \{A \cap U_n : n \in \omega\}.$$

Since each $A \cap U_n$ is residually null, we see that $A \cap R(A)$ is residually null. Moreover, as $R(A)$ is open, $\text{cl}_X(R(A)) \setminus R(A)$ is closed and nowhere dense.

Therefore,

$$A \cap \text{cl}_X(R(A)) \subseteq (A \cap R(A)) \cup (\text{cl}_X(R(A)) \setminus R(A)),$$

finishing the proof. \square

Lemma 27. Let X be a Polish space, $A, B \subseteq X$, and $A_\xi \subseteq X$ for all $\xi < \kappa$ where κ is a cardinal.

- (i) If $A \subseteq B$, then $R(B) \subseteq R(A)$ and, consequently, $CR(A) \subseteq CR(B)$;
- (ii) $R(A \cup B) = R(A) \cap R(B)$ which provides that

$$CR(A \cup B) = CR(A) \cup CR(B);$$

- (iii) $X \setminus \text{cl}_X(A) \subseteq R(A)$ which yields $CR(A) \subseteq \text{cl}_X(A)$;
- (iv) If U is open, then $CR(U) = \text{cl}_X(U)$. Moreover, $U \subseteq ICR(U)$ and both $ICR(U) \setminus U$ and $CR(U) \setminus U$ are residually null;
- (v) A is residually null if and only if $CR(A) = \emptyset$;
- (vi) $A \setminus CR(A)$ is residually null;
- (vii) $A \setminus ICR(A)$ is residually null;
- (viii) $CR(A) \setminus CR(B) \subseteq CR(A \setminus B)$;
- (ix) $CR(\bigcap\{A_\xi : \xi < \kappa\}) \subseteq \bigcap\{CR(A_\xi) : \xi < \kappa\}$;
- (x) $\bigcup\{CR(A_\xi) : \xi < \kappa\} \subseteq CR(\bigcup\{A_\xi : \xi < \kappa\})$;
- (xi) $CR(CR(A)) = CR(A)$;
- (xii) A is an \mathcal{R} -set if and only if $CR(A) \setminus A$ is residually null;
- (xiii) A is an \mathcal{R} -set if and only if $CR(A) \triangle A$ is residually null;
- (xiv) A is an \mathcal{R} -set if and only if $ICR(A) \setminus A$ is residually null;
- (xv) A is an \mathcal{R} -set if and only if $ICR(A) \triangle A$ is residually null;
- (xvi) $CR(A) = \text{cl}_X(ICR(A))$.

Proof. (i) Notice that $R(B)$ is open and $R(B) \cap A \subseteq R(B) \cap B$, the latter of which is residually null. It follows that $R(B) \subseteq R(A)$ and, therefore, that $CR(A) \subseteq CR(B)$.

(ii) From (i) we see that $R(A \cup B) \subseteq R(A) \cap R(B)$. Now, observe that

$$\begin{aligned} (A \cup B) \cap (R(A) \cap R(B)) &= (A \cap (R(A) \cap R(B))) \cup (B \cap (R(A) \cap R(B))) \\ &\subseteq (A \cap R(A)) \cup (B \cap R(B)), \end{aligned}$$

the last of which is residually null. Hence, $R(A) \cap R(B) \subseteq R(A \cup B)$. Consequently, $CR(A \cup B) = CR(A) \cup CR(B)$.

(iii) Notice that $X \setminus \text{cl}_X(A)$ is open and that $A \cap (X \setminus \text{cl}_X(A)) = \emptyset$ which is surely residually null. Hence, $X \setminus \text{cl}_X(A) \subseteq R(A)$ which provides us with the fact that $CR(A) \subseteq \text{cl}_X(A)$.

(iv) Let U be open. Since any non-empty open set is not residually null, we see that $R(U) = X \setminus \text{cl}_X(U)$. That is, $CR(U) = \text{cl}_X(U)$. Obviously, $U \subseteq \text{cl}_X(U) = CR(U)$ which implies that $U \subseteq ICR(U)$. The rest follows from the fact that $CR(U) \setminus U = \text{cl}_X(U) \setminus U$, which is closed and nowhere dense.

(v) If A is residually null, $R(A) = X$ which implies that $CR(A) = \emptyset$. If $CR(A) = \emptyset$, then $R(A) = X$ so we see that $A = R(A) \cap A$ which provides us with the fact that A is residually null.

(vi) Notice that $A \setminus CR(A) = A \cap R(A)$ which is residually null by Theorem 26.

(vii) Observe that $A \setminus ICR(A) \subseteq (A \setminus CR(A)) \cup (CR(A) \setminus ICR(A))$ and that $CR(A) \setminus ICR(A)$ is closed and nowhere dense. So (vi) guarantees that $A \setminus ICR(A)$ is residually null

(viii) Using (i) and (ii), notice that

$$\begin{aligned} CR(A) &= CR((A \setminus B) \cup (A \cap B)) \\ &= CR(A \setminus B) \cup CR(A \cap B) \\ &\subseteq CR(A \setminus B) \cup CR(B). \end{aligned}$$

Hence,

$$CR(A) \setminus CR(B) \subseteq CR(A \setminus B).$$

Both (ix) and (x) follow immediately from (i).

(xi) From (iii) we see that $CR(CR(A)) \subseteq \text{cl}_X(CR(A)) = CR(A)$. From (viii) and (vi) we see that $CR(A) \setminus CR(CR(A)) \subseteq CR(A \setminus CR(A)) = \emptyset$. Hence, $CR(A) \subseteq CR(CR(A))$ establishing that $CR(A) = CR(CR(A))$.

(xii) Surely, if $CR(A) \setminus A$ is residually null, then

$$CR(A) \triangle A = (CR(A) \setminus A) \cup (A \setminus CR(A))$$

is residually null using (vi). Since $CR(A)$ is closed, A is an \mathcal{R} -set.

Now, suppose A is an \mathcal{R} -set and, appealing to Lemma 23, write $A = (U \setminus N) \cup M$ where U is open and both N and M are residually null. Notice that $CR(U) \setminus CR(N) \subseteq CR(U \setminus N) \subseteq CR(U)$ by (viii) and (i) and that $CR(M) = CR(N) = \emptyset$ by (v). Hence, $CR(U) = CR(U \setminus N)$. It follows that, capitalizing on (ii),

$$CR(A) = CR((U \setminus N) \cup M) = CR(U).$$

Now,

$$\begin{aligned} CR(A) \setminus A &= CR(U) \setminus ((U \setminus N) \cup M) \\ &= CR(U) \cap ((U \cap N^c)^c \cap M^c) \\ &= CR(U) \cap (U^c \cup N) \cap M^c \\ &= [(CR(U) \setminus U) \cup (CR(U) \cap N)] \cap M^c \\ &\subseteq (CR(U) \setminus U) \cup N. \end{aligned}$$

By (iv), $CR(U) \setminus U$ is meager so we see that $CR(A) \setminus A$ is residually null.

(xiii) Combine (vi) and (xii).

(xiv) If $ICR(A) \setminus A$ is residually null, then

$$ICR(A) \triangle A = (ICR(A) \setminus A) \cup (A \setminus ICR(A))$$

is residually null using (vii). As $ICR(A)$ is open, A is an \mathcal{R} -set.

Now, suppose A is an \mathcal{R} -set. Then $CR(A) \setminus A$ is residually null by (xii). Since $ICR(A) \setminus A \subseteq CR(A) \setminus A$, we have that $ICR(A) \setminus A$ is residually null.

(xv) Combine (vii) and (xiv).

(xvi) Notice that, for $E \subseteq X$,

$$\begin{aligned} E \cap (\text{cl}_X(\text{int}_X(\text{cl}_X(E))))^c &\subseteq \text{cl}_X(E) \cap (\text{int}_X(\text{cl}_X(E)))^c \\ &= \text{cl}_X(E) \setminus \text{int}_X(\text{cl}_X(E)) \end{aligned}$$

and the last set is closed and nowhere dense. It follows that

$$(\text{cl}_X(\text{int}_X(\text{cl}_X(E))))^c \subseteq R(E) \implies CR(E) \subseteq \text{cl}_X(\text{int}_X(\text{cl}_X(E))).$$

Now, with (xi) and the fact that $CR(A)$ is closed,

$$CR(A) = CR(CR(A)) \subseteq \text{cl}_X(\text{int}_X(CR(A))) = \text{cl}_X(ICR(A)) \subseteq CR(A).$$

Conclusively, $\text{cl}_X(ICR(A)) = CR(A)$. \square

Corollary 28. Suppose X is Polish and that A and B are \mathcal{R} -sets. Then

$$ICR(A) \cap ICR(B) \neq \emptyset \implies A \cap B \neq \emptyset.$$

Proof. Suppose $A \cap B = \emptyset$ and notice that

$$\begin{aligned} ICR(A) \cap ICR(B) &\subseteq [(ICR(A) \setminus A) \cup A] \cap [(ICR(B) \setminus B) \cup B] \\ &\subseteq (ICR(A) \setminus A) \cup (ICR(B) \setminus B). \end{aligned}$$

Since $ICR(A) \cap ICR(B)$ is an open set and Lemma 27 (xiv) provides us with the fact that both $ICR(A) \setminus A$ and $ICR(B) \setminus B$ are residually null, we see that $ICR(A) \cap ICR(B) = \emptyset$. \square

Lemma 29. Let X be a Polish space and $A \subseteq X$. Then $A \subseteq CR(A)$ if and only if, for every open set $U \subseteq X$ with $U \cap A \neq \emptyset$, $U \cap A$ is not residually null in X .

Proof. Suppose that, for each open set $U \subseteq X$, $U \cap A \neq \emptyset$ implies that $U \cap A$ is not residually null. Then, $R(A) = X \setminus \text{cl}_X(A)$. Hence, $A \subseteq \text{cl}_X(A) \subseteq CR(A)$.

Now, assume there is an open set $U \subseteq X$ so that $U \cap A \neq \emptyset$ but $U \cap A$ is residually null. It follows that $A \cap U \subseteq U \subseteq R(A)$. For $x \in A \cap U$, $x \in A \cap R(A)$. That is, $A \not\subseteq CR(A)$. \square

A few of the results that follow generalize some of those found in [2].

Theorem 30. Suppose X is a Polish space, (Y, ρ) is a metric space, and $f : X \rightarrow Y$ satisfies the following:

- $f^{-1}[B] \in \mathcal{R}(X)$ for each open ball $B \subseteq Y$;
- $f^{-1}[V] \subseteq CR(f^{-1}[V])$ for all open $V \subseteq Y$.

Then the points of continuity of f form a dense G_δ subset of X .

Proof. Fix $n \in \omega$ and, for $x \in X$, let $V_x = B_\rho(f(x), 2^{-(n+1)})$ and $U_x = ICR(f^{-1}[V_x])$. Then define $\mathcal{U}_n = \bigcup\{U_x : x \in X\}$ and notice that \mathcal{U}_n is open. We will endeavor to show \mathcal{U}_n is dense. Let $W \subseteq X$ be a non-empty open set and pick $x \in W$. Observe that

$$x \in f^{-1}[V_x] \subseteq CR(f^{-1}[V_x]).$$

It follows that $W \cap CR(f^{-1}[V_x]) \neq \emptyset$ and is relatively open in $CR(f^{-1}[V_x])$. By Lemma 27 (xvi), we know that U_x is dense in $CR(f^{-1}[V_x])$. Hence, $W \cap U_x \neq \emptyset$. It follows that \mathcal{U}_n is open and dense in X .

Next, we'll show that $\text{osc}_f(x) \leq 2^{-n}$ at each point $x \in \mathcal{U}_n$. Towards this end, we will first see that, for any open ball $V \subseteq Y$ and $x \in ICR(f^{-1}[V])$, $f(x) \in \text{cl}_Y(V)$. Otherwise, there is an open ball W about $f(x)$ so that $W \cap V = \emptyset$. Since $x \in f^{-1}[W] \subseteq CR(f^{-1}[W])$, we see that

$$ICR(f^{-1}[V]) \cap CR(f^{-1}[W]) \neq \emptyset.$$

Alas, as $ICR(f^{-1}[W])$ is dense in $CR(f^{-1}[W])$,

$$ICR(f^{-1}[V]) \cap ICR(f^{-1}[W]) \neq \emptyset.$$

Since $f^{-1}[V]$ and $f^{-1}[W]$ are \mathcal{R} -sets, Lemma 28 implies that $f^{-1}[V] \cap f^{-1}[W] \neq \emptyset$ which provides $V \cap W \neq \emptyset$, a contradiction.

Now, let $x \in \mathcal{U}_n$ and pick $w \in X$ so that $x \in U_w$. By the above paragraph, for $y, z \in U_w$, we have that $f(y), f(z) \in \text{cl}_Y(V_w)$. It follows that

$$\rho(f(y), f(z)) \leq \rho(f(y), f(w)) + \rho(f(w), f(z)) \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}.$$

Since $x \in \mathcal{U}_n$ was arbitrary, we see that $\text{osc}_f(x) \leq 2^{-n}$ for every point $x \in \mathcal{U}_n$.

Finally, $\mathcal{U} = \bigcap\{\mathcal{U}_n : n \in \omega\}$ is a dense G_δ for which f has zero oscillation for every point of \mathcal{U} . That is, f is continuous at every point of \mathcal{U} . \square

Definition 31. Let X be a Polish space, Y be a topological space, and $f : X \rightarrow Y$ be a function. We say that f is \mathcal{R} -measurable if $f^{-1}[U]$ is an \mathcal{R} -set for every open set $U \subseteq Y$.

Theorem 32. Let X be a Polish space, Y be a topological space with a countable basis, and $f : X \rightarrow Y$ be \mathcal{R} -measurable. Then, there exists a set $A \subseteq X$ which is co-residually null so that $f \upharpoonright_A$ is continuous.

Proof. Let $\{V_n : n \in \omega\}$ be a basis for Y and notice that $f^{-1}[V_n]$ is an \mathcal{R} -set for each $n \in \omega$ by hypothesis. Hence, by Lemma 23, we can write, for each $n \in \omega$,

$$f^{-1}[V_n] = (U_n \setminus N_n) \cup M_n$$

where U_n is open and both N_n and M_n are residually null. Let

$$\begin{aligned} A &= \left[X \setminus \bigcup \{N_n : n \in \omega\} \right] \cap \left[X \setminus \bigcup \{M_n : n \in \omega\} \right] \\ &= X \setminus \left[\bigcup \{N_n : n \in \omega\} \cup \bigcup \{M_n : n \in \omega\} \right] \end{aligned}$$

and notice that A is co-residually null. Now, to see that $f \upharpoonright_A$ is continuous, notice that

$$\begin{aligned} f^{-1}[V_n] \cap A &= [(U_n \setminus N_n) \cup M_n] \cap A \\ &= (U_n \cap N_n^c \cap A) \cup (M_n \cap A) \\ &= U_n \cap A \end{aligned}$$

and $U_n \cap A$ is relatively open in A , finishing the proof. \square

Proposition 33. Let X be Polish, Y be a separable metric space, and G be a group that acts as a group of homeomorphisms on X and Y which is transitive on X . If $f : X \rightarrow Y$ is an \mathcal{R} -measurable map which is G -equivariant, then f is continuous.

Proof. First, we will see that $f^{-1}[V] \subseteq CR(f^{-1}[V])$ for any open set $V \subseteq Y$. Let $U \subseteq X$ and $V \subseteq Y$ be open so that $U \cap f^{-1}[V] \neq \emptyset$ and pick $x \in U \cap f^{-1}[V]$. Then $\langle x, f(x) \rangle \in U \times V$. For $y \in X$, let $g \in G$ be so that $g \cdot x = y$. Then

$$\langle y, f(y) \rangle = \langle g \cdot x, f(g \cdot x) \rangle = \langle g \cdot x, g \cdot f(x) \rangle \in g \cdot U \times g \cdot V.$$

It follows that $\text{graph}(f) \subseteq \bigcup \{g \cdot U \times g \cdot V : g \in G\}$. Since $\text{graph}(f)$ is separable and metrizable as it is a subset of $X \times Y$, it is Lindelöf which implies that there is $\{g_n : n \in \omega\} \subseteq G$ so that

$$\text{graph}(f) \subseteq \bigcup \{g_n \cdot U \times g_n \cdot V : n \in \omega\}.$$

Let $X_n = \{x \in X : \langle x, f(x) \rangle \in g_n \cdot U \times g_n \cdot V\} = g_n \cdot (U \cap f^{-1}[V])$. Since $X = \bigcup \{X_n : n \in \omega\}$, some X_n is not residually null. Hence, $U \cap f^{-1}[V]$ is not residually null. Thus, by Lemma 29, $f^{-1}[V] \subseteq CR(f^{-1}[V])$.

Now, by Theorem 30, we have that the points of continuity of f is a dense G_δ subset of X . To see that f is continuous everywhere, let $x \in X$. For some $y \in X$, f is continuous at y . As G is transitive, let $g \in G$ be so that $g \cdot x = y$ and $\{x_n : n \in \omega\} \subseteq X$ so that $x_n \rightarrow x$. Then $g \cdot x_n \rightarrow g \cdot x = y$. As f is G -equivariant, $f(g \cdot x_n) = g \cdot f(x_n)$ and notice that

$$g \cdot f(x_n) = f(g \cdot x_n) \rightarrow f(y) = f(g \cdot x) = g \cdot f(x)$$

as f is continuous at y . It follows that $f(x_n) \rightarrow f(x)$, which is to say that f is continuous at x , finishing the proof. \square

Corollary 34. Let G be a multiplicative group with a Polish topology so that, for each $g \in G$, the map $h \mapsto gh$, $G \rightarrow G$, is continuous and, for each

$h \in G$, the map $g \mapsto gh$, $G \rightarrow G$, is \mathcal{R} -measurable. Then G is a Polish group.

Proof. Fix $h \in G$ and let $\phi : G \rightarrow G$ be defined by $\phi(g) = gh$. By assumption, ϕ is \mathcal{R} -measurable. Let G act on itself by $\langle g, x \rangle \mapsto gx$, $G^2 \rightarrow G$, and notice that G acts as a group of homeomorphisms on G and that this action is transitive. Moreover, ϕ is G -equivariant as, for any $g, x \in G$, $\phi(gx) = gxh = g\phi(x)$. Hence, ϕ is continuous by Proposition 33. It follows that multiplication is separately continuous so, applying [11], we are done. \square

5. THE ALEXANDROV-SUSLIN OPERATION

We will now discover that \mathcal{R} -sets are closed under the Alexandrov-Suslin operation, hereinafter referred to as the \mathcal{A} -operation, which was introduced independently by P. Alexandrov in [1] and M. Suslin in [16]. One of the properties of the \mathcal{A} -operation is that all Σ_1^1 subsets of a Polish space can be obtained by the \mathcal{A} -operation on a countable family of closed sets. It's also true that countable families of Baire property sets are preserved under the \mathcal{A} -operation. Let's recall the definition.

Definition 35. Let X be a set. By an \mathcal{A} -system, we mean a function $S : \omega^{<\omega} \rightarrow \wp(X)$. For an \mathcal{A} -system S , we define the \mathcal{A} -operation on S to be

$$\mathcal{A}(S) = \bigcup_{w \in \omega^\omega} \bigcap_{n \in \omega} S(w \upharpoonright_n).$$

We say that an \mathcal{A} -system S is *regular* if $S(w \upharpoonright_{n+1}) \subseteq S(w \upharpoonright_n)$ for all $w \in \omega^{<\omega}$ and $n \in \omega$.

Lemma 36. With respect to the \mathcal{A} -operation, all \mathcal{A} -systems can be assumed to be regular without loss of generality. That is, for any \mathcal{A} -system S , there is a regular \mathcal{A} -system S' so that $\mathcal{A}(S) = \mathcal{A}(S')$.

Proof. For each $\sigma \in \omega^{<\omega}$ of length n , define

$$S'(\sigma) = \bigcap \{S(\sigma \upharpoonright_m) : m \leq n\}$$

and notice that S' is the desired system. \square

Lemma 37. Let X be a Polish space and $A \subseteq X$. There exists an \mathcal{R} -set B so that, for any \mathcal{R} -set E , if $A \subseteq E$, $B \setminus E$ is residually null. If desired, this B can be chosen to be the union of a closed set with a residually null set.

Proof. Let $F = CR(A)$, $R = (A \setminus CR(A))$, $B = F \cup R$, and notice that $A \subseteq B$. Since $A \setminus CR(A)$ is residually null by Lemma 27 (vi), Lemma 23 guarantees that B is an \mathcal{R} -set.

Now, suppose E is any \mathcal{R} -set so that $A \subseteq E$. By Lemma 23, we can write E as $(U \setminus N) \cup M$ where U is open and both N and M are residually null. Now,

$$\begin{aligned} (F \cup R) \setminus ((U \setminus N) \cup M) &= (F \cup R) \cap ((U \cap N^c)^c \cap M^c) \\ &= (F \cup R) \cap ((U^c \cup N) \cap M^c) \\ &\subseteq [F \cap ((U^c \cup N) \cap M^c)] \cup R \\ &\subseteq (F \cap U^c \cap M^c) \cup N \cup R \\ &= [F \setminus (U \cup M)] \cup N \cup R. \end{aligned}$$

So we just need to see that $F \setminus (U \cup M)$ is residually null. Observe that, appealing to Lemma 27,

$$A \subseteq U \cup M \implies CR(A) \subseteq CR(U \cup M) = CR(U) \implies F \subseteq \text{cl}_X(U).$$

Thus,

$$F \setminus (U \cup M) \subseteq \text{cl}_X(U) \setminus (U \cup M) \subseteq \text{cl}_X(U) \setminus U,$$

finishing the proof. \square

Theorem 38. The class of \mathcal{R} -sets is closed under the \mathcal{A} -operation. That is, suppose S is an \mathcal{A} -system so that, for every $\sigma \in \omega^{<\omega}$, $S(\sigma)$ is an \mathcal{R} -set. Then $\mathcal{A}(S)$ is an \mathcal{R} -set.

Proof. Apply Lemma 37 and [5, Theorem 29.13]. \square

6. APPLICATIONS TO POLISH GROUPS

Proposition 39. Let G be a multiplicative Polish group, H be a multiplicative topological group, and $\phi : G \rightarrow H$ be a group homomorphism. If there exists a set $A \subseteq X$ which is co-residually null so that $\phi \upharpoonright_A$ is continuous, then ϕ is continuous.

Proof. Let $\{g_n : n \in \omega\} \subseteq G$ be so that $g_n \rightarrow g \in G$. Since A is co-residually null, $g^{-1}A \cap \bigcap \{g_n^{-1}A : n \in \omega\}$ is co-residually null and, in particular, non-empty. So, pick $h \in g^{-1}A \cap \bigcap \{g_n^{-1}A : n \in \omega\}$. It follows that $gh \in A$ and, for each $n \in \omega$, $g_nh \in A$. Since $\phi \upharpoonright_A$ is continuous and $g_nh \rightarrow gh$,

$$\phi(g_n)\phi(h) = \phi(g_nh) \rightarrow \phi(gh) = \phi(g)\phi(h).$$

It follows that $\phi(g_n) \rightarrow \phi(g)$, which is to say that ϕ is continuous. \square

Corollary 40. Let G be a multiplicative Polish group, H be a multiplicative topological group with a countable basis, and $\phi : G \rightarrow H$ be a group homomorphism which is \mathcal{R} -measurable. Then ϕ is continuous.

Proof. Use Theorem 32 and Proposition 39. \square

Now we provide a generalization to the famous theorem of B. J. Pettis in [13].

Theorem 41. Suppose G is a multiplicative Polish group and suppose A is an \mathcal{R} -set of G which is not residually null. Then both $A^{-1}A$ and AA^{-1} contain a neighborhood of the identity.

Proof. Let U be open so that $A\Delta U$ is residually null and notice that $U \neq \emptyset$ since A is assumed to not be residually null. Pick $g \in U$ and find a neighborhood of the identity V so that $gVV^{-1} \subseteq U$. Let $h, p \in V$ and notice that $gph^{-1} \in U$ which implies that $gp \in Uh$. Also, since V is a neighborhood of identity, $gp \in gVV^{-1} \subseteq U$. As $p \in V$ was arbitrary, we see that $gV \subseteq U \cap Uh$.

For $h \in V$,

$$(U \cap Uh)\Delta(A \cap Ah) \subseteq (U\Delta A) \cup (Uh\Delta Ah) = (U\Delta A) \cup ((U\Delta A)h).$$

Since multiplication on the right is a homeomorphism and the notion of being residually null is a topological invariant by Proposition 5, we see that

$$(U \cap Uh)\Delta(A \cap Ah)$$

is residually null. Since $U \cap Uh$ is a non-empty open set, $U \cap Uh$ is not residually null by Lemma 13 so $A \cap Ah \neq \emptyset$.

So, let $h \in V$ and, since $A \cap Ah^{-1} \neq \emptyset$, let $g \in A \cap Ah^{-1}$. It follows that $g^{-1} \in A^{-1}$ and $gh \in A$. Thus, $h = g^{-1}gh \in A^{-1}A$. That is, $V \subseteq A^{-1}A$ so we see that $A^{-1}A$ contains a neighborhood of identity. A similar argument establishes the same result for AA^{-1} . \square

Corollary 42. Let G be a multiplicative Polish group and suppose H is a subgroup which is an \mathcal{R} -set but not residually null. Then H is open.

Proof. Since H is an \mathcal{R} -set which is not residually null, by Theorem 41, we see that HH^{-1} contains a neighborhood U of the identity. So $U \subseteq HH^{-1} \subseteq H$ and we see that $H = \bigcup \{hU : h \in H\}$ establishing that H is open. \square

Theorem 43. Let G be a multiplicative Polish group, H be a multiplicative separable group, and $\phi : G \rightarrow H$ be a homomorphism which is \mathcal{R} -measurable. Then ϕ is continuous.

Proof. It suffices to show that ϕ is continuous at 1_G so let U be a neighborhood of 1_H . Then pick a neighborhood V of 1_H so that $V^{-1}V \subseteq U$. Let $\{h_n : n \in \omega\} \subseteq H$ be dense and notice that, for any $h \in H$, hV^{-1} is a neighborhood of h . Pick h_n so that $h_n \in hV^{-1}$ and $p \in V^{-1}$ so that $h_n = hp$. Then $h = h_np^{-1} \in h_nV$. So we see that $H = \bigcup \{h_nV : n \in \omega\}$.

Now, $G = \bigcup \{\phi^{-1}[h_nV] : n \in \omega\}$. It follows that there must be some $n \in \omega$ so that $\phi^{-1}[h_nV]$ is not residually null. By assumption, $\phi^{-1}[h_nV]$ is

an \mathcal{R} -set so Theorem 41 provides us with the fact that

$$(\phi^{-1}[h_n V])^{-1} \phi^{-1}[h_n V]$$

contains a neighborhood of 1_G . Observe that

$$\begin{aligned} (\phi^{-1}[h_n V])^{-1} \phi^{-1}[h_n V] &= \phi^{-1}[(h_n V)^{-1}] \phi^{-1}[h_n V] \\ &\subseteq \phi^{-1}[(h_n V)^{-1} h_n V] \\ &= \phi^{-1}[V^{-1} h_n^{-1} h_n V] \\ &= \phi^{-1}[V^{-1} V] \\ &\subseteq \phi^{-1}[U], \end{aligned}$$

finishing the proof. \square

Corollary 44. Let G and H be multiplicative Polish groups and $\phi : G \rightarrow H$ be a homomorphism which is \mathcal{B} -measurable. Then ϕ is continuous. If moreover, $\phi[G]$ is not residually null, ϕ is open.

Proof. Immediately, by Theorem 43, ϕ is continuous. So suppose $\phi[G]$ is not residually null and fix an open set $W \subseteq G$ with $1_G \in W$. Let $V \subseteq G$ be open so that $1_G \in V$ and $V^{-1}V \subseteq W$. Let $\{g_n : n \in \omega\}$ be a dense subset of G and notice that $G = \bigcup \{g_n V : n \in \omega\}$. It follows that

$$\phi[G] = \phi \left[\bigcup \{g_n V : n \in \omega\} \right] = \bigcup \{\phi(g_n) \phi[V] : n \in \omega\}$$

and, since $\phi[G]$ is not residually null, there is some $n \in \omega$ so that $\phi(g_n) \phi[V]$ is not residually null. Hence, $\phi[V]$ is not residually null and, as the continuous image of an open set, $\phi[V]$ is also an analytic subset of H so $\phi[V]$ is an \mathcal{B} -set. Thus, applying Theorem 41, we see that there exists some open $W' \subseteq H$ so that

$$1_H \in W' \subseteq (\phi[V])^{-1} \phi[V] = \phi[V^{-1}] \phi[V] \subseteq \phi[V^{-1}V] \subseteq \phi[W].$$

Now, for any non-empty open set $U \subseteq G$, let $y \in \phi[U]$ and pick $x \in U$ so that $\phi(x) = y$. Find an open set W_x so that $1_G \in W_x$ and $xW_x \subseteq U$. By the above paragraph, we can find an open set $V_y \subseteq H$ so that $1_H \in V_y \subseteq \phi[W_x]$. Then

$$y \in yV_y \subseteq \phi(x) \phi[W_x] = \phi[xW_x] \subseteq \phi[U]$$

which guarantees that

$$\phi[U] = \bigcup \{yV_y : y \in \phi[U]\},$$

establishing that $\phi[U]$ is open. \square

7. MARTIN'S AXIOM AND RESIDUALLY NULL SETS

Recall that a topological space X has *countable cellularity* provided that, for any family \mathcal{U} of pair-wise disjoint open sets, $\#\mathcal{U} \leq \aleph_0$. For a set A and a cardinal κ , we will say that A is κ -sized provided $\#A \leq \kappa$.

Lemma 45. Suppose X is any topological space and $D \subseteq X$ be a dense subspace which is of countable cellularity with respect to its inherited topology. Then X is of countable cellularity.

Proof. Let \mathcal{U} be a family of pair-wise disjoint open subsets of X and define $\mathcal{U}_D = \{U \cap D : U \in \mathcal{U}\}$. Notice that \mathcal{U}_D is a family of pair-wise disjoint open subsets of D and that, since D is dense in X , $U \cap D \neq \emptyset$ for each $U \in \mathcal{U}$. As D is assumed to be of countable cellularity, we see that X is also of countable cellularity. \square

The following is the topological equivalent to the classical Martin's Axiom.

Definition 46. For a cardinal κ , let $\text{MA}(\kappa)$ be the statement: For any compact Hausdorff space X with countable cellularity and for any κ -sized collection \mathcal{U} of open dense subsets of X , $\bigcap \mathcal{U} \neq \emptyset$.

In particular, $\text{MA}(\omega)$ is true without the condition of countable cellularity and is equivalent to the Baire Category Theorem. Also, $\text{MA}(\mathfrak{c})$ is false by looking at $[0, 1]$ and, for each $x \in [0, 1]$, defining $U_x = [0, 1] \setminus \{x\}$. Then the family $\mathcal{U} := \{U_x : x \in [0, 1]\}$ is a \mathfrak{c} -sized collection of open dense subsets but $\bigcap \mathcal{U} = \emptyset$. From this, we see that Martin's Axiom is only non-trivial in models of $\neg\text{CH}$. In general, we will refer to Martin's Axiom as the assertion that $\text{MA}(\kappa)$ holds for all cardinals $\kappa < \mathfrak{c}$.

Recall that a topological space X is *Čech-complete* if X admits a compactification K in which X is a G_δ subset of K . In fact, if a topological space X is Čech-complete, X is actually a G_δ subset of any of its compactifications ([4, Theorem 3.9.1]).

Proposition 47. For any cardinal κ , $\text{MA}(\kappa)$ is equivalent to the statement: For any Čech-complete space X of countable cellularity and any κ -sized family \mathcal{U} of open dense sets, $\bigcap \mathcal{U} \neq \emptyset$;

Proof. Since any compact Hausdorff space X is trivially Čech-complete, we need only see that $\text{MA}(\kappa)$ implies the proposed equivalent.

Suppose X is a Čech-complete space of countable cellularity and let \mathcal{U} be a κ -sized family of open dense sets where $\kappa < \mathfrak{c}$. Choose a compactification K of X and notice that, by Lemma 45, K is also of countable cellularity. Then, for each $U \in \mathcal{U}$, let $V_U \subseteq K$ be open so that $X \cap V_U = U$ and define $\mathcal{V} = \{V_U : U \in \mathcal{U}\}$. As each $U \in \mathcal{U}$ is dense in X and X is dense in K , we see that each V_U is an open dense subset of K .

Now, as X is a dense G_δ in K , let \mathcal{W} be a countable family of open dense subsets of K so that $\bigcap \mathcal{W} = X$. As long as κ is an infinite cardinal, $\mathcal{W} \cup \mathcal{V}$ is a κ -sized collection of open sets so we can apply $\text{MA}(\kappa)$ to see that

$$\emptyset \neq \bigcap \mathcal{W} \cap \bigcap \mathcal{V} = X \cap \bigcap \mathcal{V} = \bigcap \mathcal{U},$$

finishing the proof. \square

Let X be Polish and recall that X can be homeomorphically embedded into $[0, 1]^\omega$. As $[0, 1]^\omega$ is also Polish, Mazurkiewicz' Theorem guarantees that X is Čech-complete and of countable singularity as X is separable.

Corollary 48. Let X be a Polish space and \mathcal{E} be a κ -sized family of meager sets where $\kappa < \mathfrak{c}$. Assuming Martin's Axiom, $\bigcup \mathcal{E}$ is meager.

Theorem 49 ([6, Exercise III.3.30]). Let X be a Polish space, μ be a Borel probability measure on X , and \mathcal{E} be a κ -sized family of μ -null sets where $\kappa < \mathfrak{c}$. Assuming Martin's Axiom, $\bigcup \mathcal{E}$ is μ -null.

Theorem 50. Let X be a Polish space and \mathcal{E} be a κ -sized family of residually null sets where $\kappa < \mathfrak{c}$. Assuming Martin's Axiom, $\bigcup \mathcal{E}$ is residually null.

Proof. Let $\mathcal{E} = \{E_\xi : \xi < \kappa\}$ be a collection of residually null sets and notice that, by Corollary 48, $\bigcap \{N(E_\xi) : \xi < \kappa\}$ is co-meager in $\mathcal{M}(X)$. Let $\mu \in \bigcap \{N(E_\xi) : \xi < \kappa\}$ and notice that, by Theorem 49, $\mu(\bigcup \mathcal{E}) = 0$. Hence, $\bigcap \{N(E_\xi) : \xi < \kappa\} \subseteq N(\bigcup \mathcal{E})$ which affirms that $\bigcup \mathcal{E}$ is residually null. \square

Corollary 51. Let X be a Polish space and \mathcal{E} be a κ -sized family of \mathcal{R} -sets where $\kappa < \mathfrak{c}$. Assuming Martin's Axiom, $\bigcup \mathcal{E}$ is an \mathcal{R} -set.

Proof. Let $\mathcal{E} = \{E_\xi : \xi < \kappa\}$ be a collection of \mathcal{R} -sets and pick U_ξ open so that $E_\xi \triangle U_\xi$ is residually null for each $\xi < \kappa$. Then, observe that

$$\left[\bigcup \{E_\xi : \xi < \kappa\} \right] \triangle \left[\bigcup \{U_\xi : \xi < \kappa\} \right] \subseteq \bigcup \{E_\xi \triangle U_\xi : \xi < \kappa\}$$

which, by appealing to Theorem 50, completes the proof. \square

Recall that the *density* of a topological space X is the least cardinal κ so that there exists a κ -sized dense subset of X . We will let $\text{den}(X)$ denote the density of X .

Corollary 52. Let G be a Polish group, H be a topological group with $\text{den}(H) < \mathfrak{c}$, and $\phi : G \rightarrow H$ be a homomorphism which is \mathcal{R} -measurable. Assuming Martin's Axiom, ϕ is continuous.

Proof. In the proof of Theorem 43, replace the dense set $\{h_n : n \in \omega\}$ with a dense set $\{h_\xi : \xi < \kappa\}$. Then use Theorem 50 to see that there has to be some $\xi < \kappa$ so that $\phi^{-1}[h_\xi V]$ is not residually null. \square

REFERENCES

- [1] P. S. Alexandrov, *Sur la puissance des ensembles mesurables B*, C.R. Acad. Sci. Paris **162** (1916), 323–325.

- [2] Michael P. Cohen and Robert R. Kallman, *A conjecture of Gleason on the foundations of geometry*, Topology Appl. **161** (2014), 279–289.
- [3] Lester Dubins and David Freedman, *Measurable sets of measures*, Pacific J. Math. **14** (1964), no. 4, 1211–1222.
- [4] Ryszard Engelking, *General Topology*, Sigma series in pure mathematics, vol. 6, Heldermann Verlag, 1989.
- [5] Alexander S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, 1995.
- [6] Kenneth Kunen, *Set Theory*, Studies in Logic, vol. 43, College Publications, 2011.
- [7] K. Kuratowski, *Topology*, vol. 1, Academic Press, 1966.
- [8] Paul Larson, Itay Neeman, and Saharon Shelah, *Universally measurable sets in generic extensions*, Fund. Math. **208** (2010), 173–192.
- [9] R. Daniel Mauldin, David Preiss, and Heinrich v. Weizsacker, *Orthogonal transition kernels*, Ann. Probab. **11** (1983), no. 4, 970–988.
- [10] Arnold W. Miller, *Special subsets of the real line*, Handbook of Set-Theoretic Topology (Kenneth Kunen and Jerry E. Vaughan, eds.), North Holland, 1984, pp. 201–233.
- [11] Deane Montgomery, *Continuity in topological groups*, Bull. Amer. Math. Soc. **42** (1936), no. 12, 879–882.
- [12] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, AMS Chelsea Publishing Series, Academic Press, 1967.
- [13] B. J. Pettis, *On continuity and openness of homomorphisms in topological groups*, Ann. Math. **52** (1950), no. 2, 293–308.
- [14] Saharon Shelah, *Can you take Solovay’s inaccessible away?*, Israel J. Math. **48** (1984), 1–47.
- [15] Robert M. Solovay, *A model of set-theory in which every set of reals is Lebesgue measurable*, Ann. Math. **92** (1970), no. 1, 1–56.
- [16] M. Ya. Suslin, *Sur un définition des ensembles mesurables B sans nombres transfinis*, C.R. Acad. Sci. Paris **164** (1917), 88–91.